

# Kinetic roughening in two-phase fluid flow through a random Hele-Shaw cell

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A nonlocal interface equation is derived for two-phase fluid flow, with arbitrary wettability and viscosity contrast  $c = (\mu_1 - \mu_2)/(\mu_1 + \mu_2)$ , in a model porous medium defined as a Hele-Shaw cell with random gap  $b_0 + \delta b$ . Fluctuations of both capillary and viscous pressure are explicitly related to the microscopic quenched disorder, yielding conserved, non-conserved and power-law correlated noise terms. Two length scales are identified that control the possible scaling regimes and which scale with capillary number as  $\ell_1 \sim b_0(cCa)^{-1/2}$  and  $\ell_2 \sim b_0Ca^{-1}$ . Exponents for forced fluid invasion are obtained from numerical simulation and compared with recent experiments.

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The displacement of a fluid by another in a porous medium is a problem of fundamental interest in nonequilibrium physics as a paradigm of interface dynamics in disordered media [1, 2]. Experiments on bead packs in Hele-Shaw cells [3] in particular, have stimulated considerable theoretical efforts, but the problem has consistently revealed itself rather elusive [1, 2]. More recently a new surge of interest has arisen with the recognition of the inherently non-local character of the problem as a key ingredient [4, 5], and the realization of a new series of experiments in Hele-Shaw cells with random gap [6, 7, 8]. Roughening exponents of the proposed nonlocal equations have been explored by means of Flory-type scaling arguments [4] and phase field simulations [5, 6]. While the specific properties of noise are known to be crucial to determine the universal aspects of interface roughening, fluctuations are usually modeled at a phenomenological level, and including only local capillary effects. Noise related to the non-Laplacian viscous pressure due to quenched disorder in the permeability has been so far neglected. While this may be justified for imbibition experiments [5], other situations, such as forced fluid invasion, do require a quantitative assessment of this point. In addition, it would be desirable to have a unified formulation for general conditions of viscosity contrast  $c = (\mu_1 - \mu_2)/(\mu_1 + \mu_2)$  and wettability given the rich variety of phenomena that the experimental evidence has unveiled as a function of those parameters [2].

Here we address the general problem of fluid displacement in a Hele-Shaw cell with random gap, as a simple model of a porous medium. This model system has the great advantage that no coarse-graining procedure must be invoked in the theoretical formulation, thus allowing us to derive *ab initio* a general and complete interface equation, quantitatively accurate, with explicit dependence on 'bare' parameters, and including all noise sources. On the experimental side, the system is also appealing since a direct control of the disorder is locally possible on the microscopic scale [6, 7].

A complete description of interface fluctuations must contain three basic physical effects of a porous matrix on

its motion, namely local variations of (i) capillary pressure, (ii) permeability, and (iii) available volume. Different but not independent noise terms must thus be generated through distinct physical mechanisms from the unique source of randomness. Fluctuating capillary pressure can be directly related to gap variations  $\delta b = b - b_0$  where  $b_0$  is the mean value, in terms of the Young-Laplace condition for the pressure jump across the interface

$$p_2 - p_1 = \sigma \left( \kappa + \frac{2 \cos \theta}{b_0 + \delta b} \right) \quad (1)$$

where  $\kappa$  is the curvature in the cell plane,  $\sigma$  is surface tension and  $\theta$  is the contact angle,  $\cos \theta = 1$  meaning perfect wetting of the (invading) fluid 1. The effect of viscous pressure fluctuations, however, is far less obvious due to the inherently nonlocal character of the interplay of (ii) and (iii) in the response of the fluid flow to gap fluctuations. We base our analysis on the assumption that, for sufficiently smooth gap variation (i.e.  $|\nabla b| \ll 1$ ) Darcy's law for a Hele-Shaw cell [9] holds locally as

$$\mathbf{v} = - \frac{[b_0 + \delta b(x, y)]^2}{12\mu} \nabla p. \quad (2)$$

In a capillary tube of lateral size  $b$  at fixed injection pressure, from Darcy's law a relative velocity fluctuation scales as  $\delta v/v \sim 2\delta b/b$  (larger permeability implies less resistance to flow) while at fixed flow injection it is exactly the opposite,  $\delta v/v \sim -2\delta b/b$  (mass conservation slows down the flow if there is more volume available). In an actual disordered medium the solution of the whole pressure field will thus be required to sort out the effective flow conditions at each location. A direct consequence of (iii) is that the 2d effective flow, must give rise in general to a non-conserved interface equation, precisely to account for volume conservation in 3d. In our case, 3d incompressibility implies that the 2d flow will satisfy

$$\nabla \cdot (b\mathbf{v}) = 0, \quad (3)$$

where the gap acts effectively as a variable density.

Eqs. (2) and (3) imply that the pressure field is non-Laplacian. In order to obtain a closed bulk equation which we can project onto the interface degrees of freedom we treat the pressure perturbatively in  $|\nabla b|$ . We thus split the pressure field as  $p = p_0 + \delta p$  and keep only the order  $|\nabla b|$  in  $\delta p$  to obtain

$$\nabla^2 p_0 = 0 \quad (4a)$$

$$\nabla^2 \delta p + \frac{3\nabla b}{b} \cdot \nabla p_0 = 0, \quad (4b)$$

where we have neglected higher orders consistently with the fact that they have also been omitted in assuming local Darcy flow. The lowest order Eq. (4a) can be solved as the unperturbed problem [9, 10] with the modified boundary condition Eq. (1) which contains all capillary effects, while the simplest boundary condition  $\delta p = 0$  is then required for the Poisson Equation (4b). In terms of the Laplacian Green's function  $\delta p$  then satisfies

$$\int_{int} ds G(x - x(s), y - y(s)) \frac{\partial \delta p}{\partial n} = - \int dx' dy' G(x - x', y - y') \frac{3\nabla b}{b} \cdot \nabla p_0 \quad (5)$$

The free-boundary problem is thus defined by Eq. (2) specified at the interface, with  $p = p_0 + \delta p$ , and the boundary conditions at infinity. Here we focus on the case of forced fluid invasion, where a fixed velocity  $V$  is imposed at infinity and  $\mu_1 \geq \mu_2$ . We introduce the dimensionless quenched noise as  $b^2 = b_0^2[1 + \xi(x, y)]$ . Noise originated respectively from  $\delta b$  in Eq. (1), from  $\delta b$  in Eq. (2) and from  $\delta p$  in Eq. (4b) will be called respectively *capillary*, *permeability* and *bulk* noises.

Concerning the scaling properties of the interface, we are interested in the lowest order approximation on the interface deviation from planarity, which is relevant in a Renormalization Group (RG) sense. Our result for the interface equation in Fourier space for two-fluid displacement under constant injection velocity  $V$  takes the form

$$\frac{1}{V} \frac{\partial \hat{h}(k)}{\partial t} = \delta(k) - c|k| [1 + (\ell_1 k)^2] \hat{h}(k) + \mathcal{N}_h(k) - \frac{1}{2} (1 + \ell_2 |k|) \hat{\xi}_h(k) + \hat{\Omega}_{LR}(k, t) \quad (6)$$

where the lengths  $\ell_1$  and  $\ell_2$  are defined in terms of the capillary number  $\text{Ca} = 12(\mu_1 + \mu_2)V/\sigma$  as  $\ell_1 = b_0(c\text{Ca})^{-1/2}$  and  $\ell_2 = 2b_0 \cos \theta \text{Ca}^{-1}$ , and where  $\mathcal{N}_h(k)$  denotes the leading (quadratic) nonlinearities

$$\mathcal{N}_h(k) = -c|k| \int_{-\infty}^{\infty} dq [1 - \mathcal{S}(kq)] |q| [1 + (\ell_1 q)^2] \hat{h}(k - q) \hat{h}(q) \quad (7)$$

where  $\mathcal{S}$  is the sign function.  $\hat{\xi}_h(k)$  is the Fourier transform of  $\xi(x, h(x))$  and the term  $\hat{\Omega}_{LR}(k, t)$  is a long-ranged

correlated noise of the form

$$\hat{\Omega}_{LR}(k, t) = \frac{1}{\mu_1 + \mu_2} [\mu_1 \hat{\Omega}^-(k, t) + \mu_2 \hat{\Omega}^+(k, t)] \quad (8)$$

with

$$\hat{\Omega}^{\pm}(k, t) = |k| \int dx dy \xi(x, y + Vt) e^{-ikx \mp y|k|} \Theta(\pm y), \quad (9)$$

where  $\Theta$  is the step function.

Note that the long-ranged term  $\hat{\Omega}_{LR}(k, t)$  enters effectively as an annealed (explicitly time-dependent) noise (see discussion below). Eqs. (6–9) constitute our central result. Note also that in this formulation we have assumed weak noise so that multiplicative noise terms of order  $h\xi$  or nonlinear in  $\xi$  have been neglected [13].

The linear deterministic part of Eq. (6) is well known [9, 10]. The complete set of deterministic nonlinearities can be obtained systematically using the weakly nonlinear expansion developed in Ref. [10]. These include the familiar local terms considered in Ref. [4], such as  $(\nabla h)^2$ , but also nonlocal terms in real space.

The capillary fluctuations give rise to the conserved (area-preserving) noise term proportional to  $|k|\hat{\xi}_h$ . This contribution is associated to the second term in Eq. (1) and is generated exactly as the usual capillary term  $-|k|^3 \hat{h}(k)$  which comes from the first term in Eq. (1) [9, 10]. A conserved noise term of this form was phenomenologically argued in Ref. [5]. On the other hand, the non-conserved noise term proportional to  $\hat{\xi}_h$  results from the trivial lowest order contribution of permeability noise of the form  $V\hat{\xi}_h(k)$ , plus a nontrivial local term of opposite sign coming from the expression of the bulk noise defined as  $\delta v_{\xi} \approx -\frac{b_0^2}{12\mu} \frac{\partial \delta p}{\partial n}$  which takes the form

$$\widehat{\delta v_{\xi}}(k) = \frac{3V}{2} (-\hat{\xi}_h(k) + \hat{\Omega}_{LR}(k, t)). \quad (10)$$

We now proceed to sketch the derivation of the bulk noise. For simplicity we will consider the one-sided case ( $c = 1$ ). Our derivation for  $c \neq 1$  follows the formulation of Ref. [9] but is more involved and will be presented elsewhere [11]. Neglecting orders  $\xi \partial_x h$ , Eq. (5) reads

$$\begin{aligned} \frac{-2}{3} \int_{-\infty}^{\infty} dx' \ln [(x - x')^2 + (h(x) - h(x'))^2] \delta v_{\xi}(x') = \\ \int_{-\infty}^{\infty} dx' \left\{ \ln [(x - x')^2 + (h(x) - h(x'))^2] \xi(x', h(x')) \right. \\ \left. + \int_{-\infty}^{h(x')} dy' \frac{2(h(x') - y')}{(x - x')^2 + (h(x) - y')^2} \xi(x', y') \right\} \quad (11) \end{aligned}$$

where the integral on  $y$  of the rhs of Eq. (5) has been integrated by parts and it has been assumed that the noise vanishes at infinity,  $\xi(x, y \rightarrow -\infty) = 0$ . It has also been applied that  $\frac{\nabla b}{b} \nabla p_0 \simeq -\frac{6\mu}{b_0^2} \frac{\partial \xi}{\partial y} V$ , and that the Green function has the form  $G(x, y) = -(4\pi)^{-1} \ln[x^2 +$

$y^2]$ . With the change of integration  $y = y' - Vt$ , and to lowest order in the interface deviation  $h - Vt$  with respect

to the mean interface position [14], Eq. (11) then reads

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$$\int_{-\infty}^{\infty} dx' \ln |x - x'| \delta v_{\xi}(x') = -\frac{3}{2} \int_{-\infty}^{\infty} dx' \left\{ \ln |x - x'| \xi(x', h) + \int_{-\infty}^0 dy \frac{-y}{(x - x')^2 + y^2} \xi(x', y + Vt) \right\}. \quad (12)$$


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Eq. (12) can be explicitly solved for  $\delta v_{\xi}$  using distribution Fourier calculus [12] to yield the result Eq. (10).

The quenched noise will be typically correlated on a microscopic scale  $a$ . For  $ka \ll 1$ ,  $\xi$  is effectively white. If  $\langle \xi(x, y) \xi(x', y') \rangle = \Delta \delta(x - x') \delta(y - y')$  then we have

$$\langle \hat{\Omega}_{LR}(k, t) \hat{\Omega}_{LR}(k', t') \rangle = \frac{\Delta}{2\pi} |k| \delta(k + k') e^{-|k|V|t-t'|} \quad (13)$$

so  $\hat{\Omega}_{LR}$  scales as  $|k|^{1/2}$  and introduces long-range memory. Accordingly, low- $k$  behavior is dominated by the local part of bulk noise. This implies that 3d conservation overcomes permeability at low- $k$ , giving rise to an overall non-conserved noise with the same sign as the capillary noise. Although direct computation of both terms in the bulk noise shows that the local part is typically larger, it is unclear to what extent neglecting the long-range term may miss important details of local interface pinning which may eventually affect the scaling. Furthermore, for  $ka \sim 1$ , both local and nonlocal parts of the bulk noise are comparable but then the annealing approximation may not be justified, and a more careful analysis of bulk noise based in Eq. (11) may be necessary [11]. The local and non-local contributions to bulk noise may be of the same order in other situations. For instance, persistent noise  $\xi = \xi(x)$  yields  $\delta v_{\xi} = 0$ , with non-conserved and conserved noises opposing each other.

One of the salient features of Eq. (6) is that the problem has two characteristic length scales.  $\ell_1$  controls the well-known crossover between (deterministic) capillary and viscous forces [5]. The second length  $\ell_2$  is a newly identified one which defines a crossover between conserved and non-conserved noise. For general viscosities and wetting conditions the two length scales are arbitrary and define a variety of possible scaling regimes and crossovers depending on their relative size. Experiments of a wetting fluid invading an inviscid one ( $c = 1$ ,  $\cos \theta = 1$ ) typically have  $\ell_2 \gg \ell_1$ . Concerning the nonlinearities, power counting arguments show that quadratic terms are relevant in the RG sense only if the viscous term  $c|k|$  is absent, that is, for  $|k|\ell_1 \gg 1$ . This capillary-dominated regime will always be observed at large  $k$  and short times provided  $\ell_2 \gg \ell_1$ . A crossover to the viscous-dominated regime will eventually occur for  $|k|\ell_1 \sim 1$  and then the quadratic nonlinearities do become irrelevant. A second crossover within the viscous regime, from conserved to non-conserved noise, will occur for  $|k|\ell_2 \sim 1$ .

The explicit knowledge of the bare coefficients of the nonlinear terms is helpful to assess the validity of the linear approximation. Numerical simulation of typical cases show that the effect of nonlinearities in the capillary regime is not appreciable in reasonable simulation times. The capillary regime, which is relevant to experiments [8], is thus well described by

$$\frac{1}{V} \frac{\partial \hat{h}(k)}{\partial t} = \delta(k) - |k| \left[ (\tilde{\ell}_1 k)^2 \hat{h}(k) + \frac{1}{2} \ell_2 \hat{\xi}_h(k) \right] \quad (14)$$

with  $\tilde{\ell}_1 = b_0 \text{Ca}^{-1/2}$ . Since the quadratic nonlinearities vanish for  $c = 0$  (for symmetry), we may refer to the growth described by Eq. (14), as the universality class of ‘symmetric fluid invasion’ (SFI) [15].

We will now study the SFI scaling by numerical simulation of Eq. (14) and also the subsequent crossovers. The quantities of interest concerning the scaling properties are the root mean square of the interface height fluctuations,  $W$ , and the structure factor  $S(k, t)$ . The width grows with time as  $W(t) \sim t^{\beta}$  before saturation, and the saturation width scales with system size  $L$  as  $W_{sat} \sim L^{\alpha}$ . The roughness exponent  $\alpha$  can also be obtained from the relation  $S(k, t) \sim k^{-1-2\alpha}$  for long times. For further reference, we provide the scaling exponents of Eq. (14) for two cases that are exactly solvable [11], namely, for persistent noise  $\xi = \xi(x)$ , and annealed noise  $\xi = \xi(x, t)$  which, assuming  $\delta$ -correlations, give respectively  $\alpha = 3/2$ ,  $\beta = 1/2$  and  $\alpha = 0$ ,  $\beta = 0$ .

Typical results for Eq. (14) are shown in Figs. 1 and 2. We choose  $V = 1$ ,  $a = 0.0625$ ,  $\ell_1 = 50$  and  $\ell_2 = 3000$ , which satisfy the criterion  $a \ll \ell_1 \ll \ell_2$ . For these values, we have obtained a roughness exponent  $\alpha = 1.2 \pm 0.05$  (very close to the value reported in Ref. [5]) and a growth exponent  $\beta = 0.68 \pm 0.02$ . The scaling of the correlation function  $G(l, t)$  has been found to be fully consistent with  $\alpha_{loc} = 1$ , as expected from the superrough value of  $\alpha > 1$ . Note that the scaling of the power spectrum  $S(k, t)$  at large  $k$  corresponds to the case of persistent noise,  $\alpha = 3/2$ : short segments of the interface ‘feel’ effectively a persistent noise for long enough time intervals.

An increase (by an order of magnitude) in the value of  $V$  with the subsequent variation of  $\ell_1$  and  $\ell_2$  modifies the scaling behavior: for low values of  $k$  the scaling of  $S(k, t)$  yields  $\alpha = 0$ , the value obtained for annealed noise, while for larger  $k$  the observed scaling is essentially the same

as in Fig. 2. Hence, the effectively noise acting on the interface at low- $k$  for large  $V$  is annealed. The value of the exponent  $\beta$  is observed to decrease with increasing  $V$ , consistently with the trend towards an effective annealed noise, for which  $\beta = 0$ . On the other hand, if  $V$  is decreased by an order of magnitude, the scaling observed in Figs. 1 and 2 is essentially unchanged.

The values of  $Ca$  and  $V$  we have used are of similar magnitude to the ones of the experimental work of Ref. [8], and the exponents reported there for large injection rates,  $\alpha \simeq 0$  for small  $k$  and  $\alpha \simeq 1.3$  for large  $k$  are fully consistent with our results for large  $V$ . However, this is not the case for small values of  $V$ . This discrepancy could be attributed to non-darcyan effects associated to the sharp edges in the gap variations of the experiment, which may modify pinning properties.

After the first crossover to the viscous regime with conserved noise, we find that the interface is not rough,  $W_{sat}(L \rightarrow \infty) = const.$  After the second crossover to non-conserved noise, we get  $\alpha = \beta = 0$ , with  $W$  growing logarithmically both with time  $t$  and system size  $L$ .

As a final remark let us mention that the case of spontaneous imbibition (constant pressure conditions) can be described with our formalism with an appropriate time-dependence  $V(t) \sim t^{-1/2}$ , except close to pinning, when the zero-mode fluctuations must be carefully worked out.

We expect that the physical insights and the predictive power of the theoretical framework here presented may be useful to reinterpret data and design new experiments, in particular in the yet unexplored range of parameters. The extent to which it may be applicable to more realistic porous media, with appropriate coarse-grained parameters, however, is an open question which deserves further study.

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  - [13] Terms of this kind coming from capillary and permeability can be included systematically [11], but those from the bulk cannot be obtained explicitly.
  - [14] Unlike local noise terms, the ‘annealed’ approximation is justified in  $\delta v_\xi$  as a leading order, because of its integral form. This means that the noise acting on a region where the interface is pinned does vary because the interface is moving elsewhere and it couples through the bulk.
  - [15] With respect to RG flow of  $c$  this is a saddle fixed point.

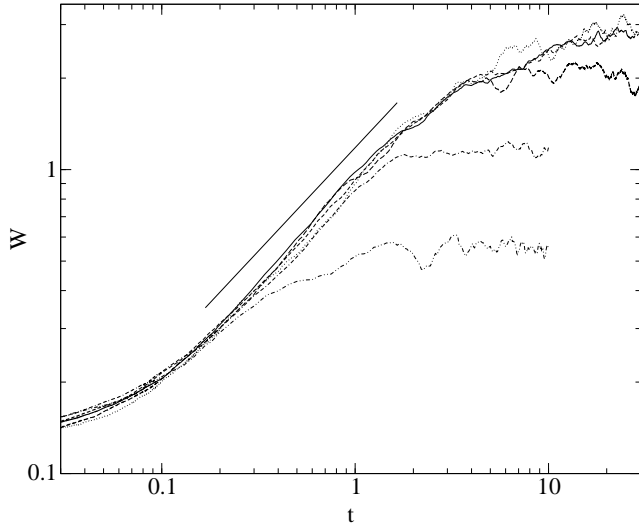


FIG. 1: Interface width  $W$  as a function of time, for system sizes  $L = 32, 64, 128, 256, 512$  and  $1024$ . The straight line is a fit with a slope  $\beta = 0.68$ .

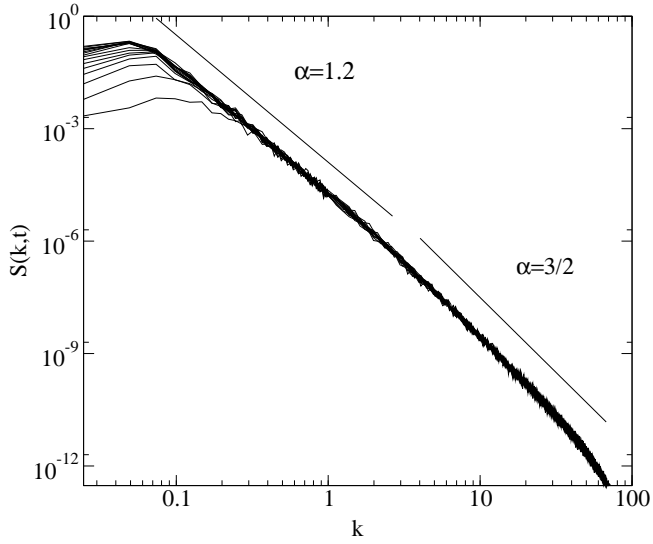


FIG. 2: Structure factor for a system with  $L = 256$ . The data are for  $t = 0.5$  (lower curve) to  $t = 12.0$ , and time interval  $\Delta t = 0.5$ . The straight line with slope  $-3.3$  ( $\alpha = 1.2$ ) is a fit, and the other straight line has slope  $-4$  ( $\alpha = 3/2$ ).